# LOGIC

## Statement

In everyday speech, in written text, we have sentences that are true or false,sentences that are logical corect.

Those sentences in mathematic we call statements. **The statement is any sentence you knows that it can only be correct or only incorrect.** In other words, a statement can only have one of the two values: true (correct), false (incorrect).

Example:

It is not difficult to see which of the following sentence are statements:

- Number 6 is greater than the number 2
- Number 3 is divisible by number 2
- Earth revolves around the Sun
- Issue 2 is greater than Natasa
- Year has 365 days

The first three sentences are statements, because they are correct or incorrect, while the last two sentence we can not say, therefore,they are not statements.

**Statements will, by agreement, note with : p, q, r, s, t. ...**

From such fundamental statements we create complex statements.

Will be important for us to know when new statements will be correct or incorrect.

We will use:

**T – for correct and $\perp$-for incorrect**

**$\tau$(t)=T, if a statement t is correct**

**$\tau$(t)=$\perp$, if a statement t is incorrect**

## LOGICAL OPERATIONS

**Conjunction (and)**of statement p and q is statement $p \wedge q$ which corresponds to the following table:

| p | q | $p \wedge q$ |
|---|---|---|
| **T** | **T** | **T** |
| T | ⊥ | ⊥ |
| ⊥ | T | ⊥ |
| ⊥ | ⊥ | ⊥ |

**It is true if and only if both of its operands are true.**

**Disjunction (or)** of statement p and q is statement $p \vee q$ :

| p | q | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | ⊥ | T |
| ⊥ | T | T |
| ⊥ | ⊥ | ⊥ |

**It is false only if both of its operands are false.**

You need to pay attention to the difference between disjunction and exclusive disjunction which corresponds to the language form “**or .. or .... ".**

The difference is that the statement “or p or q "is not correct in the case when both statements are true, while the statement "p or q”in this case is correct.

**Imlication (if…then)** of statement p and q is statement $p \Rightarrow q$ :

| p | q | $p \Rightarrow q$ |
|---|---|---|
| T | T | T |
| **T** | ⊥ | ⊥ |
| ⊥ | T | T |
| ⊥ | ⊥ | T |

**It is false just in case the first operand is true and the second operand is false.**

One of the most important "payer" for us is implication. Sentence **p implies q** with unchanged meaning can be record at one of the following ways:

- "If p, then q"
- "From p follows q "
- "q, if p"
- "p is a sufficient condition for q "
- "q is necessary condition for the p"

**Equivalence (if and only if)** is statement $p \Leftrightarrow q$ :

| p | q | $p \Leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | ⊥ | ⊥ |
| ⊥ | T | ⊥ |
| ⊥ | ⊥ | T |

**equivalence is correct only if both statements are true or both statements are false.**

Sentence "**p is equivalent to q " we can** say at one of the following ways:

- "If p, then q, and if q, then p"
- "p is necessary and sufficient condition for q "
- " p if and only if q"

These were the basic **binary** logic operations. Binary, because the two statements made a new statement.

**Unary** operation is $\neg$ (not).

**Negation of** statement p is statement $\neg p$ which corresponds to table:

| P | $\neg p$ |
|---|---|
| T | $\perp$ |
| $\perp$ | T |

**Negation is operation of the highest priority!**Then conjunction and disjunction, which are each equal, at the end are implication and equivalence, also mutually equal.

**Example:**

A series of symbol $p \wedge q \vee r$ **we can not accept** as a formula, because we do not know the order of operations!

This should be written in he following manner: $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$.

Therefore, the statements of the representing formula established by letters p, q, r, ..., characters $\wedge, \vee, \Rightarrow, \Leftrightarrow, \neg$, and brackets, using the final number of times these symbols.

**Formulas that are true, for all possible values of letters that make the formula, we called a tautology.**

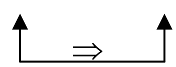
Whether the formula is tautology we can check in several ways: the discussion by letter, decrease the discrepancy ,over the table of value, etc..

Here's one example of testing over  table:

F: $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$

**Take heed: Negation is  “oldest operations”**

| p | q | $\neg q$ | $\neg p$ | $\neg q \Rightarrow \neg p$ | $p \Rightarrow q$ | $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$ |
|---|---|---|---|---|---|---|
| T | T | $\bot$ | $\bot$ | T | T | T |
| T | $\bot$ | T | $\bot$ | $\bot$ | $\bot$ | T |
| $\bot$ | T | $\bot$ | T | T | T | T |
| $\bot$ | $\bot$ | T | T | T | T | T |




## Some rules of  logical conclusion

$p \wedge (p \Rightarrow q) \Rightarrow q$   modus ponens

$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$   contraposition rule (examined in the table)

$(\neg p \Rightarrow (q \wedge \neg q)) \Rightarrow p$   decrease in contradiction

$p \vee \neg p$   law out of the third

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$   De Morgan laws
$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

$\neg\neg\, p \Leftrightarrow p$   dual negation law, etc..

**modus ponens by the use of the contradiction:**

$p \wedge (p \Rightarrow q) \Rightarrow q$

**Suppose that the formula is incorrect.** Since the implications is incorrect only in one case, it must be:

$\tau(p \wedge (p \Rightarrow q)) = \mathrm{T}$ and $\tau(q) = \perp$

Further, as conjunction is correct only if both terms are correct, it must be:

$\tau(p) = \mathrm{T}$ and $\tau(p \Rightarrow q) = \mathrm{T}$

Let we see table for implication:

| p | q | p⇒q |
|---|---|---|
| T | T | T |
| **T** | ⊥ | ⊥ |
| ⊥ | T | T |
| ⊥ | ⊥ | T |

As we have concluded that : $\tau(p) = \mathrm{T}$ and $\tau(q) = \perp$ it must be $\tau(p \Rightarrow q) = \perp$, which is in contradiction with $\tau(p \Rightarrow q) = \mathrm{T}$.

**This means that starting assumption is not good, and that the formula is tautology (correct).**

**Example:**

**Examine whether the following formula tautology:**

F: $(\neg p \Rightarrow q) \Leftrightarrow ((r \vee p) \wedge q)$

When we have three of the representing letters, we need 8 types:

| **p** | **q** | **r** | $\neg p$ | $\neg p \Rightarrow$ **q** | **r** $\vee$ **p** | **(r** $\vee$ **p)** $\wedge$ **q** | **F** |
|---|---|---|---|---|---|---|---|
| T | T | T | $\perp$ | T | T | T | T |
| T | T | $\perp$ | $\perp$ | T | T | T | T |
| T | $\perp$ | T | $\perp$ | T | T | $\perp$ | $\perp$ |
| T | $\perp$ | $\perp$ | $\perp$ | T | $\perp$ | $\perp$ | $\perp$ |
| $\perp$ | T | T | T | T | T | T | T |
| $\perp$ | T | $\perp$ | T | T | T | T | T |
| $\perp$ | $\perp$ | T | T | $\perp$ | T | $\perp$ | T |
| $\perp$ | $\perp$ | $\perp$ | T | $\perp$ | $\perp$ | $\perp$ | T |

**So this formula is not tautology, because all values are not true.**

## Quantification

**Look the sentence: "** $\mathbf{x}^2 = 25$ ".Obviously, it **is not** a statement, as it may be correct if x = 5 or x =-5, and may be invalid if x is some other number.

If, however, we say a sentence:

"For each **x,** $\mathbf{x}^2 = 25$" or "There is x, $\mathbf{x}^2 = 25$" then for them, we can say that first is incorrect and the second is correct, and **they are statements**.

Using mathematical terminology we can write:

"For each **x,** $\mathbf{x}^2 = 25$" ⟶ $(\forall x)\ (\mathbf{x}^2 = 25)$

"There is x, $\mathbf{x}^2 = 25$" ⟶ $(\exists x)\ \ (\mathbf{x}^2 = 25)$

We use the universal quantifier symbol $\forall$ ( an upside – down letter "A") to indicate **universal quantification**.

$\forall$ - we read : "for all"; "for each"

$\exists$ - we read : 'there exists', and called **existential** quantification.

It is interesting how quants **act** in **the presence of negation:**

$$\neg(\forall x)A \Leftrightarrow (\exists x)\neg A \quad \text{and}$$

$$\neg(\exists x)A \Leftrightarrow (\forall x)\neg A$$

For example, the sentence: "Not every professor is good" and "There is the professor who is not good" have the same meaning.